ON THE PARABOLIC REGULARITY, SOBOLEV EMBEDDINGS AND GLOBAL CARLEMAN ESTIMATES IN $L^q(L^p)$ SPACES

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ABSTRACT. In this paper we discuss some aspects related to regularity in parabolic problems with corollaries regarding anisotropic Sobolev embeddings. We use these results in the context of bootstrap arguments applied to global Carleman estimates for nonhomogeneous parabolic equations in $L_t^q(L_x^p)$ spaces, estimates which are fundamental in associated control and inverse problems.

The arguments we use are characterizations of regularity in terms of domains of fractional powers of elliptic operators and then characterization of these domains as interpolation spaces and relations to Bessel potential and Sobolev-Slobodeckii spaces.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with smooth boundary $\partial\Omega$ and denote by $Q = \Omega \times]0, T[$. We consider parabolic problems of the form

$$(1.1) \quad \begin{cases} D_t y(t,x) + Ly(t,x) = f(t,x) & t \in]0,T[, x \in \Omega, \\ y(t,x) = 0 & t \in]0,T[, x \in \partial\Omega, \\ y(0,x) = y_0(x) & x \in \Omega, \end{cases}$$

where L is an uniformly elliptic operator of the form

$$(1.2) \quad Ly = -\sum_{j,k=1}^n D_j(a_{jk}D_k y) + \sum_{k=1}^n b_k D_k y + cy.$$

The coefficients satisfy the regularity assumptions $a_{jk} \in W^{1,\infty}(\Omega)$, $b^k, c \in L^\infty(\Omega)$ and those in principal part satisfy for some $\mu > 0$ the ellipticity condition

$$(1.3) \quad \sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k \geq \mu|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, x \in \Omega.$$

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Other boundary conditions which are naturally associated to operator L may also be considered in order to study the problem in an abstract form as

(1.4) $y'(t) + Ay(t) = f(t), y(0) = y_0,$

where $-A$ is generator of an analytic semigroup $S(t) = e^{-tA}$, in a Banach space X, $f \in L^q(X) := L^q(0, T; X)$, $y_0 \in X$. The mild solution is written in the form

(1.5) $y(t) = e^{-tA}y_0 + \int_0^t e^{-(t-s)A} f(s)ds = S(t)y_0 + [S * f](t).$

The classical reference for existence and regularity of solutions to parabolic problems with $f \in L^p(Q) = L^p(L^p(\Omega))$ is the monograph by O.A.Ladyzenskaja, V.A.Solonnikov, N.N.Uralceva [20], where maximal regularity is obtained in the anisotropic Sobolev spaces $W_p^{2,1}(Q)$. The regularity of solutions to abstract parabolic problems (1.4) by using the representation (1.5) and estimates in real interpolation spaces, was considered in the paper of Gabriella Di Blasio [13]; there, for $f \in L^q(X)$ one obtains $S*f \in W^{\theta,q}(X), \theta \in]0, 1[$ and $S * f \in L^q(D_A(\theta, p)), \theta \in]0, 1[$ (here $D_A(\theta, p) = (X, D(A))_{\theta,p}$ and $W^{\theta,q}(X)$ is a vector valued Sobolev-Slobodeckii space).

The existence and maximal regularity in concrete parabolic problems with $X = L^p(\Omega)$ is established by W.von Wahl in [34], where estimates for $S*f \in L^q(D(A)), D_t(S*f) \in L^q(X)$ in terms of norm of $f \in L^q(L^p(\Omega))$ with $q, p > 1$ are obtained by applying a refined study of A.Benedek, A.P.Calderón, R.Panzone [7] on the convolution of operators, using ideas from the theory of singular integrals.

When dealing with parabolic problems with nonhomogeneous boundary conditions, a study of maximal regularity in $L^q(L^p)$ spaces was established by P.Weidemaier [35, 36].

Maximal regularity in $L^q(X)$ for abstract parabolic problem (1.4) is deeply related to the geometry of X and properties of operator A. More precisely, if X is UMD space (a space with unconditional martingale difference property or, equivalently, a space having the property that the vector valued Hilbert transform is bounded in $L^q(X)$) and A is sectorial with bounded imaginary powers, $A \in BIP(X, \theta)$, with spectral angle $\theta < \frac{\pi}{2}$, then equation (1.4) has maximal regularity property: $y \in W^{1,q}(X) \cap L^q(D(A))$. We refer here to the monograph of C.Martinez Carracedo, M. Sanz Alix [23], Chapter 8 and the references therein. An essential ingredient in the approach of such problems is a theorem of G.Dore and A.Venni characterizing invertibility of sums of operators in BIP class.

The BIP class is important in our presentation of parabolic regularity; it allows to characterize the domains of powers of positive operators defined by elliptic operators with boundary conditions, as complex interpolation spaces. In such situation these are closed subspaces of Bessel potential spaces (this important result is due to R.T.Seeley [29]). Then, by using

an argument based on extension operators one may relate these spaces to Sobolev-Slobodeckii spaces. Another ingredient we use in studying regularity is represented by convolution estimates in $L^r(D(A^\gamma))$, by using estimates which are specific to analytic semigroups, in domains of fractional powers of the generating operator.

As we are interested in L^p realizations of elliptic operators in bounded domains, we mention that the boundedness of imaginary powers of such operators was proved by R.T.Seeley in [31] by using a representation of the resolvent and the theory of pseudodifferential operators ([30, 28]). A more direct approach to such results was given by J.Prüss and H.Sohr in [26] (see also Th. 12.1.12 in [23]). We also mention here the paper by R.Denk, G.Dore, M.Hieber, J.Prüss, A.Venni[12] for a study of elliptic operators with Hölder coefficients in principal part, in connection to the $\mathcal{H}^\infty$ calculus and the BIP property.

The parabolic regularity we present may be derived from existing theory in the cited literature and we chose to present it in a more concentrated appearance, which is useful for studying regularity in nonlinear parabolic problems, through bootstrap arguments, when the nonlinearity depends on y and Dy. The parabolic regularity results are then used to present alternative proofs to classical embeddings for anisotropic Sobolev spaces and we also use this approach to Sobolev embeddings of $W_{p,q}^{2,1}(Q)$ spaces.

Concernig classical Gagliardo-Nirenberg inequalities for Sobolev-Slobodeckii spaces, in the most general framework, we refer to the papers of H.Brezis and P.Mironescu [9],[10]. We discuss Gagliardo-Nirenberg type inequalities for anisotropic Sobolev spaces.

Global Carleman inequalities in L^2 for parabolic problems were established by O.Yu.Imanuvilov in the context of controllability problems when the control is supported in a subdomain. We refer to the work of A.V.Fursikov and O.Yu.Imanuvilov[17] and the monograph of V.Barbu [6]; see also the paper of E.Fernandez-Cara, E.Zuazua [16] where the cost of approximate controllability is estimated through a careful analysis of the Carleman inequalities and the constants there involved. Global parabolic Carleman estimates in $L^q, q \leq 2$ for homogeneous parabolic equations, in the context of control and observability, were considered by V.Barbu [5].

These estimates found applications to the stability estimates in inverse parabolic problems. We refer to the work of O.Yu.Imanuvilov and M.Yamamoto [19] for L^2 stability. The L^q Carleman estimates in the framework of inverse problems were studied by E.A.Melnig in [24]. This motivates us to apply our regularity arguments to establish global Carleman parabolic estimates in $L^q(L^p)$, $q, p > 2$ spaces, for nonhomogeneous parabolic equations.

2. Function spaces and Sobolev embeddings

Interpolation. Consider two Banach spaces E_0, E_1 with contionuous and dense embedding $E_1 \subset E_0$. For $\theta \in]0, 1[$ denote by $[E_0, E_1]_\theta$ the complex

interpolation space of order θ. If $p \in]1, \infty[$, denote by $(E_0, E_1)_{\theta,p}$ the real interpolation space. When E_0, E_1 are Hilbert spaces, $[E_0, E_1]_\theta = (E_0, E_1)_{\theta,2}$.

If F_0, F_1 are another two Banach spaces with contionuous and dense embedding $F_1 \subset F_0$ and $T \in L(E_0, F_0)$ and $T \in L(E_1, F_1)$ then the Riesz-Thorin-Marcinkiewicz theorem states that $T \in L([E_0, E_1]_\theta, [F_0, F_1]_\theta)$ with convex inequality

$$\|T\|_{L([E_0,E_1]_\theta,[F_0,F_1]_\theta)} \leq \|T\|_{L(E_0,F_0)}^{1-\theta} \|T\|_{L(E_1,F_1)}^{\theta}.$$

Spaces of functions. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. The classical Sobolev spaces of integer order k and for $p \in [1, \infty[$ are

$$(2.1) \qquad W^{k,p}(\Omega) = \{f \in L^p(\Omega) : D^\beta f \in L^p(\Omega), \text{ for } |\beta| \leq k\},$$

where $\beta = (\beta_1, \ldots, \beta_n)$ is a multi-index, $|\beta| = \beta_1 + \cdots \beta_n$, and the associated norm is the canonical one.

For a given $s \in]0, 1[$ and $p \in [1, \infty[$ the Sobolev-Slobodeckii space (or fractional Sobolev space) is defined by

$$(2.2)$$
$$W^{s,p}(\Omega) = \left\{ f \in L^p(\Omega) : \|f\|_{L^p} + \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{1/p} < \infty \right\},$$

with the norm $\|f\|_{W^{s,p}(\Omega)}$ the finite quantity above. For $s > 1$ not an integer the space $W^{s,p}(\Omega)$ is defined as the space of functions $f \in W^{[s],p}(\Omega)$ such that $D^\beta f \in W^{s-[s],p}(\Omega)$ for all multi-index β with $|\beta| = [s]$ and is endowed with the natural norm.

For the fractional Sobolev spaces $W^{s,p}(\Omega)$, with $s \in]0, 1[$ and $p \in]1, \infty[$, one has a characterization by real interpolation (see [33], p.317):

$$(2.3) \qquad\qquad W^{s,p}(\Omega) = (L^p(\Omega), W^{1,p}(\Omega))_{s,p}.$$

The Bessel potential spaces for $s > 0, p \in [1, \infty[$ are defined using the Fourier transform $\mathcal{F}$ as

$$(2.4)$$
$$H^{s,p}(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) : \|f\|_{H^{s,p}} = \|\mathcal{F}^{-1}[(1+|x|^2)^{s/2}\mathcal{F}f]\|_{L^p} < \infty\}.$$

For $s = k \in \mathbb{N}$ and $p \in]1, \infty[$ one has

$$H^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n).$$

If $p = 2$ this is an immediate consequence of properties of the Fourier transform. However, in the case $p \neq 2$ this is a deep result and the proof relies on a study of the Bessel potential using the theory of singular integrals (see [32], Theorem 3,p.135).

The Bessel potential spaces behave well under complex interpolation, defining a complete scale of spaces (see [33], p.185): if $s = \theta s_2 + (1 - \theta)s_1$ for some $\theta \in]0, 1[$ then

$$(2.5) \qquad\qquad H^{s,p}(\mathbb{R}^n) = [H^{s_1,p}(\mathbb{R}^n), H^{s_2,p}(\mathbb{R}^n)]_\theta.$$

One defines $H^{s,p}(\Omega)$ as the space of restrictions to Ω of functions in $H^{s,p}(\mathbb{R}^n)$ with $\|f\|_{H^{s,p}(\Omega)} = \inf\{\|g\|_{H^{s,p}(\mathbb{R}^n)} : g|_\Omega = f\}$.

Consider also for $k \in \mathbb{N}$, $C^k(\overline{\Omega})$ to be the space of functions $f : \Omega \to \mathbb{R}$ for which $D^\beta f$ is bounded and uniformly continuous on Ω, for $0 \le |\beta| \le k$. For $0 < \alpha < 1$, the Hölder space $C^{k,\alpha}(\overline{\Omega})$ is the subspace of $C^k(\overline{\Omega})$ of functions f for which

$$\sup_{x,y\in\Omega, x\neq y} \frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\alpha} < +\infty.$$

$C^{k,\alpha}(\overline{\Omega})$ is a Banach space with the norm
(2.6)

$$\|f\|_{C^{k,\alpha}(\overline{\Omega})} := \max_{0\le|\beta|\le k} \sup_{x\in\Omega} |D^\beta f(x)| + \max_{0\le|\beta|\le k} \sup_{x,y\in\Omega, x\neq y} \frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\beta}.$$

We will also consider anisotropic Hölder spaces in $\overline{Q}$, $C^{\alpha, \frac{\alpha}{2}}(\overline{Q})$, with α not integer. For $0 < \alpha < 1$, this is the space of continuous functions $g : \overline{Q} \to \mathbb{R}$ having the property that $t \to g(t,x)$ is in $C^{0,\frac{\alpha}{2}}([0,T])$ for all $x \in \overline{\Omega}$, with uniformly bounded norm with respect to $x \in \overline{\Omega}$, and $x \to g(t,x)$ belongs to $C^{0,\alpha}(\overline{\Omega})$ for all $t \in [0,T]$, with uniformly bounded norm with respect to $t \in [0,T]$. It is a Banach space with the norm

$$(2.7) \qquad \|g\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q})} := \sup_{(t,x)\in\overline{Q}} |g(t,x)| + \sup_{(t,x)\neq(s,y)\in\overline{Q}} \frac{|g(t,x) - g(t,y)|}{(|x-y|^2 + |t-s|)^{\frac{\alpha}{2}}}$$

For noninteger $\alpha > 1$ we refer to [20] for the definition of $C^{\alpha,\frac{\alpha}{2}}(\overline{Q})$.

In the present paper we will use the classical results on embeddings of Bessel potential and Sobolev spaces into Lebesgue and Hölder spaces (see *e.g.* [1],[8],[3],[33]) which state that

Theorem 2.1. *For $0 < s$ and $1 < p < \infty$ the following continuous embeddings hold:*

$$W^{s,p}(\Omega), H^{s,p}(\Omega) \subset \begin{cases} L^{\tilde{p}}(\Omega), \ \tilde{p} = \frac{np}{n-sp} & if \ sp < n, \\ L^{\tilde{p}}(\Omega), \ \tilde{p} \in [p,\infty[& if \ sp = n, \\ C^{r,\alpha}(\overline{\Omega}), \ \alpha \in]0,1[, r \in \mathbb{N}, r+\alpha = s - \frac{n}{p} & if \ sp > n. \end{cases}$$

Let $(X, \|\cdot\|)$ be a Banach space. One defines the vector valued Lebesgue spaces, Sobolev spaces and Hölder spaces in analogy to the corrsponding scalar cases (by replacing $|\cdot|$ with $\|\cdot\|$ when this is applied to functions) and we denote by $L^q(X) := L^q(0,T;X), W^{s,q}(X) := W^{s,q}(0,T;X), C^{k,\alpha}(X) = C^{k,\alpha}([0,T];X)$.

We will be particularly interested in the case $X = L^p(\Omega)$ for some $p \in]1,\infty[$. In estimates in the last section we will denote, for simplicity, by $L^q(L^p)$ the space $L^q(L^p(\Omega))$.

One defines the anisotropic Sobolev spaces as:

$$(2.8) \qquad\qquad W_p^{2,1}(Q) = W^{1,p}(L^p(\Omega)) \cap L^p(W^{2,p}(\Omega)).$$

For the study of parabolic problems in these spaces, we refer to the classical monograph [20]. We give here a simplified version of Lemma 3.3 from the cited reference, which states corresponding Sobolev embeddings:

Theorem 2.2. *Consider $u \in W_p^{2,1}(Q)$, $p \in]1, \infty[$. Then $u \in Z_1$, with*

$$Z_1 = \begin{cases} L^r(Q) & \text{with } r = \frac{(n+2)p}{n+2-2p} & \text{when } p < \frac{n+2}{2} \\ L^r(Q) & \text{with } r \in [p, \infty[, & \text{when } p = \frac{n+2}{2} \\ C^{\alpha, \alpha/2}(\overline{Q}) & \text{with } 0 < \alpha < 2 - \frac{n+2}{p}, & \text{when } p > \frac{n+2}{2} \end{cases}$$

and there exists $C = C(Q, p, n)$ such that

$$\|u\|_{Z_1} \leq C \|u\|_{W_p^{2,1}(Q)}.$$

Moreover, $Du \in Z_2$ with

$$Z_2 = \begin{cases} L^{r_1}(Q) & \text{with } r_1 = \frac{(n+2)p}{n+2-p} & \text{when } p < n+2 \\ L^{r_1}(Q) & \text{with } r_1 \in [p, \infty[, & \text{when } p = n+2 \\ C^{\alpha, \alpha/2}(\overline{Q}) & \text{with } 0 < \alpha < 1 - \frac{n+2}{p}, & \text{when } p > n+2 \end{cases}$$

and there exists $C = C(Q, p, n)$ such that

$$\|Du\|_{Z_2} \leq C \|u\|_{W_p^{2,1}(Q)}.$$

For $p, q \in [1, \infty[$ consider the spaces (see [35, 36]):

$$W_{p,q}^{2,1}(Q) = L^q(W^{2,p}(\Omega)) \cap W^{1,q}(L^p(\Omega)).$$

One of the main results in the paper is about Sobolev type embeddings for $W_{p,q}^{2,1}(Q)$, and the approach will rely on the regularity of flows generated by analytic semigroups.

3. Operators, semigroups and parabolic regularity

We recall a number of classical notions and results from the Theory of operator semigroups, for which we refer *e.g.* to [25, 22]. We also discuss convolution estimates between operator valued and vector valued functions.

Analytic semigroups. Let X be a Banach space and $A : D(A) \subset X \to X$ be a positive operator such that $-A$ is the generator of an analytic semigroup on X. This is equivalent to the existence of some $\delta > 0, \frac{\pi}{2} > \omega > 0$ such that

$$\sigma(A) \subset V_{\delta, \omega} := \{\lambda \in \mathbb{C} : |\arg(\lambda - \delta)| < \omega\}$$

and for some constant $M > 0$ and $\lambda \in \mathbb{C} \setminus V_{\delta, \omega}$ one has the estimate for the resolvent $R(\lambda, A) = (\lambda I - A)^{-1}$:

$$\|R(\lambda, A)\| \leq \frac{M}{1 + |\lambda|}.$$

Denoting by $S(t) = e^{-tA}$ the semigroup generated by $-A$ one may consider the linear nonhomogeneous Cauchy problem in X, (1.4), whose mild solution is given by (1.5).

Powers of operators and domains. The fractional powers of A are defined for $\alpha \in \mathbb{C}, Re\,\alpha < 0$ as:

$$A^\alpha = \frac{1}{2\pi} \int_{\gamma_{r,\theta}} \lambda^\alpha R(\lambda, A) d\lambda,$$

where $\gamma_{r,\theta} = \{re^{i\theta} : r \geq 0\} \cup \{re^{-i\theta} : r \geq 0\}$ oriented from $\infty e^{i\theta}$ to $\infty e^{-i\theta}$ and $\omega < \theta < \pi$. For $Re\,\alpha \in]-1, 0[$ this formula may be put in the form:

$$A^\alpha x = -\frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^\alpha (\lambda I + A)^{-1} x d\lambda.$$

For α, β with negative real part one has $A^{\alpha+\beta} = A^\alpha A^\beta$. For $Re\,\alpha < -n$ one has $\mathrm{Range}A^\alpha \subset D(A^n)$ and $A^n A^\alpha = A^{n+\alpha}$. This motivates the definition of the powers for $0 \leq Re\,\alpha < n, n \in \mathbb{N}$ as

$$D(A^\alpha) = \{x \in X : A^{\alpha-n} x \in D(A^n)\}, A^\alpha x = A^n A^{\alpha-n} x.$$

We note here that a fundamental property, characterising the norm in domains of fractional powers of the trajectories, of the semigroup generated by $-A$ is that (see [25], Th.6.13) for $\alpha \in]0, 1]$, there exists $M = M(\alpha)$ such that on $]0, T]$ one has for all $x \in X$:

$$(3.1) \qquad \|A^\alpha S(t)x\| \leq Mt^{-\alpha}\|x\|.$$

We are interested in our paper on characterisations of domains of fractional powers of positive operators, generating analytic semigroups, as above, and we have (see [33] Th.1.15.3, [22] Th.4.2.6, [23] Th.11.6.1):

Theorem 3.1. *If A is a densely defined positive operator, such that $A^{it} \in L(X), t \in \mathbb{R}$, then for $0 \leq Re\,\alpha < Re\,\beta$ one has*

$$(3.2) \qquad [D(A^\alpha), D(A^\beta)]_\theta = D(A^{(1-\theta)\alpha+\theta\beta}).$$

A more direct approach to the characterisation of L^p realizations of elliptic operators as belonging to the BIP class comes through transference methods (see [11]) and we mention in this respect the following theorem (see [22], Th.4.2.4):

Theorem 3.2. *Let (Ω, μ) be a $\sigma-$finite measure space and let $p \in]1, \infty[$. Suppose A is a positive operator in $L^p(\Omega, \mu)$ such that for $\lambda > 0$, $\|(\lambda I + A)^{-1}\| < \frac{1}{\lambda}$ and $(\lambda I + A)^{-1}$ has the property that $(\lambda I + A)^{-1}f \geq 0$ in Ω, whenever $f \in L^p(\Omega, \mu)$ and $f \geq 0$ in Ω. Then A has bounded imaginary powers in $L^p(\Omega, \mu)$.*

We consider now an uniformly elliptic operator in the form (1.2) with the assumed regularity for coefficients and ellipticity conditions. The L^p realization for some $p \in]1, \infty[$, with homogeneous Dirichlet boundary conditions for L takes into account the L^p regularity theory for elliptic equations (see [18]) and is defined as $A = A_p : D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ with $Au = Lu$, $u \in D(A)$.

Without loss of generality concerning regularity we may suppose that L is positive. In fact there always exists $\lambda_0 > 0$ such that $L + \lambda_0 I$ is positive. Moreover, one knows that $-A$ generates an analytic semigroup in L^p.

Maximum principle applied to elliptic operator L shows that $(\lambda I + A)^{-1}$ is positivity preserving and by Theorem 3.2 we find that $A = A_p$ has bounded imaginary powers. We have thus:

Theorem 3.3. *The operator $A = A_p$ with $p \in]1, \infty[$, which is the L^p realization of elliptic operator L with homogeneous boundary conditions on $\partial\Omega$, has the property that, for $\gamma \in]0, 1[$,*

$$D(A^\gamma) = [L^p(\Omega), W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)]_\gamma.$$

The characterization of complex interpolation spaces between domains of operators with boundary conditions is studied in [29]. The result in our case of Dirichlet homogeneous boundary conditions tells basically that $[L^p(\Omega), W^{2,p} \cap W_0^{1,p}(\Omega)]_\gamma$ coincides with $H^{2\gamma,p}(\Omega)$ for $2\gamma p < 1$ and is the closed subspace of $H^{2\gamma,p}(\Omega)$ containing functions with null boundary conditions when $2\gamma p \geq 1$ (in the case $2\gamma p = 1$ the trace is understood in a generalized sense). Relation between domains of fractional powers of operator A and Sobolev-Slobodeckii spaces is given in the next proposition:

Proposition 3.4. *Consider $\gamma \in]0, 1[$. Then, if $p \geq 2$, one has the continuous inclusion*

$$D(A^\gamma) \subset H^{2\gamma,p}(\Omega) \subset W^{2\gamma,p}(\Omega).$$

If $1 < p < 2$ and $\gamma' < \gamma$,

$$D(A^\gamma) \subset H^{2\gamma,p}(\Omega) \subset W^{2\gamma',p}(\Omega).$$

with continuous inclusion.

Proof. Consider a linear continuous extension operator

$$E : W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) + L^p(\Omega) \to W^{2,p}(\mathbb{R}^n) + L^p(\mathbb{R}^n).$$

By Riesz-Thorin-Marcinkiewicz we have that E is continuous between interpolated spaces

$$E \in L([X, D(A)]_\gamma, [L^p(\mathbb{R}^n), H^{2,p}(\mathbb{R}^n)]_\gamma)$$

Observe that, if $p \geq 2$, one has by Theorem 5, p.155 in [32]

$$H^{2\gamma,p}(\mathbb{R}^n) = [L^p(\mathbb{R}^n), H^{2,p}(\mathbb{R}^n)]_\gamma \subset W^{2\gamma,p}(\mathbb{R}^n).$$

By the same theorem in [32], if $1 < p < 2$

$$H^{2\gamma,p}(\mathbb{R}^n) = [L^p(\mathbb{R}^n), H^{2,p}(\mathbb{R}^n)]_\gamma \subset B_{p,2}^{2\gamma}(\mathbb{R}^n) =$$

$$= (L^p(\mathbb{R}^n), W^{2,p}(\mathbb{R}^n))_{\gamma,2} \subset (L^p(\mathbb{R}^n), W^{2,p}(\mathbb{R}^n))_{\gamma',p} = W^{2\gamma',p}(\mathbb{R}^n).$$

Here $B_{p,2}^{2\gamma}$ is a Besov space (see [33]). On the other hand,

$$D(A^\gamma) = [X, D(A)]_\gamma.$$

Suppose first that $p \geq 2$ and $u \in D(A^\gamma)$; we have

$$\|u\|_{D(A^\gamma)} \geq C\|Eu\|_{[L^p, H^{2,p}]_\gamma} \geq C\|Eu\|_{W^{2\gamma,p}(\mathbb{R}^n)} \geq C\|u\|_{W^{2\gamma,p}(\Omega)}.$$

If $1 < p \leq 2$ and $u \in D(A^\gamma)$ and $0 < \gamma' < \gamma$, we have by a similar argument

$$\|u\|_{D(A^\gamma)} \geq C\|Eu\|_{[L^p, H^{2,p}]_\gamma} \geq C\|Eu\|_{W^{2\gamma',p}(\mathbb{R}^n)} \geq C\|u\|_{W^{2\gamma',p}(\Omega)}.$$

Last inequalities above follow from the fact that Sobolev-Slobodeckii norm is in integral form and restricting the respective integrals to Ω will decrease the value of the integral. □

Convolution estimates. When searching for various types of solutions (classical, strong, weak) one needs to study convolutions between operator valued functions and vector valued functions.

Besides the spaces $L^q(X)$, we also consider the space $L^q_w(X) = L^q_w(0, T; X)$ for $q > 1$, which is defined, analogously to the scalar case, as the space of measurable functions $h : [0, T] \to X$ such that

$$(3.3) \qquad \|h\|_{L^q_w(X)} := \sup_{\lambda > 0} \lambda \mu(\{t \in [0, T] : \|h(t)\|_X \geq \lambda\})^{\frac{1}{q}} < \infty.$$

We may formulate the following result on convolutions of operator valued with vector valued functions, analogously to the scalar case:

Lemma 3.5. *Let X, Y be two Banach spaces and $p, q, r \in [1, \infty]$ such that $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Consider $S \in L^p(0, T; L(X, Y))$ and $f \in L^q(0, T; X)$ Then $S * f$ defined by $[S * f](t) = \int_0^t S(t - s)f(s)ds$ belongs to $L^r(0, T; Y)$ and*

$$(3.4) \qquad \|S * f\|_{L^r(Y)} \leq \|S\|_{L^p(L(X,Y))}\|f\|_{L^q(X)}.$$

Moreover, if $p, q, r \in]1, \infty[$ and $S \in L^p_w(L(X, Y))$, then $[S * f](t) = \int_0^t S(t - s)f(s)ds$ belongs to $L^r(0, T; Y)$ and there exists $C = C(p, q)$ such that

$$(3.5) \qquad \|S * f\|_{L^r(Y)} \leq C\|S\|_{L^p_w(L(X,Y))}\|f\|_{L^q(X)}.$$

Proof. The proof of this result follows easily from the result in the scalar case. Since $\|[S * f](t)\|_Y \leq \int_0^t \|S(t - s)\|_{L(X,Y)}\|f(s)\|_X ds$, we may apply the convolution estimates for scalar functions $\|S(\cdot)\|_{L(X,Y)}$ and $\|f(\cdot)\|$, for which we refer to Lemma 1.4 respectively to Theorem 1.5 in [4]. □

Regularity for parabolic problems. Consider $X = L^p(\Omega)$ and the parabolic problem with homogeneous initial data:

$$(3.6) \qquad y' + Ay = f, y(0) = 0, t \in]0, T[$$

with A the L^p realization of parabolic operator L with Dirichlet boundary conditions. It turns out that $D(A) = W^{2,p} \cap W_0^{1,p}(\Omega)$. The mild solution is given by

$$(3.7) \qquad y(t) = \int_0^t e^{-(t-s)A}f(s)ds.$$

Our purpose is to obtain regularity in $L^r(D(A^\gamma))$ and, subsequently, relating $D(A^\gamma)$ to Bessel potential and Sobolev-Slobodeckii spaces, in $L^r(H^{s,p}(\Omega))$ and $L^r(W^{s,p}(\Omega))$, for some $s > 0, r > 1$.

Spatial regularity.

Theorem 3.6. *Let A be a positive operator with $-A$ generator of an analytic semigroup, $q \in]1, \infty[$ and $f \in L^q(X)$. Consider $\gamma \in [\frac{q-1}{q}, 1[$. With $r \in]q, \infty]$ such that $1 + \frac{1}{r} = \frac{1}{q} + \gamma$, we have $y \in L^r(D(A^\gamma))$ (for $\gamma = \frac{q-1}{q}$ we have $y \in L^\infty(D(A^{\frac{q-1}{q}})))$. Moreover, for some constant $C = C(\gamma, q)$*

$$(3.8) \qquad \|y\|_{L^r(D(A^\gamma))} \leq C\|f\|_{L^q(X)}.$$

If for some $\sigma \in]0, 1[$ we have $f \in L^q(D(A^\sigma))$, then for $\sigma + \frac{q-1}{q} \leq \gamma < 1 + \sigma$ and r given by $1 + \frac{1}{r} = \frac{1}{q} + \gamma - \sigma$ one has $y \in L^r(D(A^\gamma))$. Moreover, for r, γ, σ as above there exists $C = C(\sigma, \gamma, q)$ such that

$$(3.9) \qquad \|y\|_{L^r(D(A^\gamma))} \leq C\|f\|_{L^q(D(A^\sigma))}.$$

Proof. By Theorem 6.13, [25], p.74 ,

$$\|A^\gamma S(t)\|_{L(X)} \leq \frac{C}{t^\gamma}$$

This implies that for $1 \leq \tilde{q} < \frac{1}{\gamma}$

$$\|S(\cdot)\|_{L(X, D(A^\gamma))} \in L^{\tilde{q}}(0, T),$$

and in the limit

$$\|S(\cdot)\|_{L(X, D(A^\gamma))} \in L_w^{\frac{1}{\gamma}}(0, T).$$

Then, by convolution estimates we have:

$$\|y\|_{L^r(D(A^\gamma))} \leq C\|S(\cdot)\|_{L_w^{\frac{1}{\gamma}}(L(X, D(A^\gamma)))}\|f\|_{L^q(X)} \leq C\|f\|_{L^q(X)}.$$

The case $f \in L^q(D(A^\sigma))$ may be treated in the same way by using convolution estimates and taking into account that, for $\gamma > \sigma$, there exists $C = C(\sigma, \gamma)$ such that

$$\|A^\gamma S(t)x\|_{L(X)} \leq \frac{C}{t^{\gamma-\sigma}}\|A^\sigma x\|, \; x \in D(A^\sigma),$$

which gives that for $\sigma + \frac{q-1}{q} \leq \gamma < 1 + \sigma$ one has

$$S \in L_w^{\frac{1}{\gamma-\sigma}}(L(D(A^\sigma), D(A^\gamma)))$$

and (3.9) follows as above.

$\square$

Temporal regularity. The next result is probably known but we did not find the appropriate reference. It is a generalization of Th.3.1,Ch.4 in [25], with similar ideea of proof, and gives regularity in spaces $C^{0,\alpha}(D(A^\gamma))$ for appropriate $\alpha, \gamma \in]0, 1[$.

Theorem 3.7. *For $f \in L^q(X), q \in]1, \infty[$, the mild solution $y = S * f$ given by (3.7) belongs to the spaces $C^{0, \frac{q-1}{q}-\gamma}(D(A^\gamma))$, $\gamma \in]0, \frac{q-1}{q}[$, and there exists $C = C(q, \gamma)$ such that the following estimate holds*

$$\|S * f\|_{C^{0, \frac{q-1}{q}-\gamma}(D(A^\gamma))} \leq C\|f\|_{L^q(X)}.$$

Proof. First, observe that $\|A^\gamma S(t)x\|_X \leq C\frac{1}{t^\gamma}\|x\|_X$ and $\frac{1}{t^\gamma} \in L^{q'}(0, T)$, $q' = \frac{q}{q-1}$, as $q'\gamma < 1$. From Lemma 3.5 we have a first estimate

$$(3.10) \qquad \qquad \|S * f\|_{L^\infty(D(A^\gamma))} \leq C\|f\|_{L^q(X)}.$$

In order to prove Hölder continuity, we need now the following classical estimate for the analytic semigroup generated by $-A$ (see [25], Th.6.13,Ch.2):

$$\|S(t)x - x\|_X \leq C(\alpha)t^\alpha\|A^\alpha x\|_X, x \in D(A^\alpha), \alpha \in]0, 1],$$

which has as consequence,

$$\|A^\gamma S(t + h)x - A^\gamma S(t)x\|_X \leq C(\alpha, \gamma)\frac{h^\alpha}{t^{\alpha+\gamma}}\|x\|_X, 0 < \alpha, \gamma, \alpha + \gamma < 1.$$

Now we estimate the difference

$$A^\gamma y(t + h) - A^\gamma y(t) =$$

$$= \int_t^{t+h} A^\gamma S(t+h-s)f(s)ds + \int_0^t A^\gamma[S(t+h-s) - S(t-s)]f(s)ds = I_1 + I_2.$$

We estimate the two terms:

$$\|I_1\|_X \leq \int_t^{t+h} \|A^\gamma S(t + h - s)f(s)\|_X ds = \int_0^h \|A^\gamma S(\tau)f(t + h - \tau)\|_X d\tau \leq$$

$$\leq \int_0^h \frac{1}{\tau^\gamma}\|f(t + h - \tau)\|_X d\tau \leq \left(\int_0^h \|f(t + h - \tau)\|_X^q d\tau\right)^{\frac{1}{q}} \left(\int_0^h \frac{1}{\tau^{\gamma q'}}d\tau\right)^{\frac{1}{q'}} \leq$$

$$(3.11) \qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq C\|f\|_{L^q(X)}h^{\frac{1}{q'}-\gamma},$$

with $q' = \frac{q}{q-1}$ and $\gamma q' < 1$.

For I_2 we choose $\alpha + \gamma = \frac{1}{q'}$ and we have

$$\|I_2\|_X \leq \int_0^t \|A^\gamma[S(t+h-s) - S(t-s)]\|_{L(X,X)} \|f(s)\|_X ds \leq$$

$$\leq \int_0^t C(\alpha,\gamma) \frac{h^\alpha}{(t-s)^{\alpha+\gamma}} \|f(s)\|_X ds \leq$$

$$(3.12) \qquad\qquad \leq C(\alpha,\gamma) h^\alpha \left\| \frac{1}{(\cdot)^{\alpha+\gamma}} * \|f(\cdot)\|_X \right\|_{L^\infty(0,T)} \leq$$

$$\leq C(\alpha,\gamma) h^\alpha \left\| \frac{1}{(\cdot)^{\alpha+\gamma}} \right\|_{L_w^{\frac{1}{\alpha+\gamma}}(0,T)} \|f\|_{L^q(X)}.$$

Conclusion now follows from (3.11),(3.12) and (3.10).

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Consequences of spatial regularity. Proposition 3.4 and Theorem 3.6 have as immediate consequence

Proposition 3.8. *Consider $q, p \in]1, \infty[$ and $f \in L^q(L^p(\Omega))$. For $r \in]q, \infty]$ and $\theta = 2 + \frac{2}{r} - \frac{2}{q}$, the mild solution y to (3.6), given by (3.7), satisfies the regularity estimate:*

$$(3.13) \qquad\qquad \|y\|_{L^r(H^{\theta,p}(\Omega)))} \leq C\|f\|_{L^q(L^p(\Omega))},$$

with a constant $C = C(p,q,r)$.

Moreover, for $r_1 \in]q, \infty[$ if $q \geq 2$ and $r_1 \in]q, \frac{2q}{2-q}]$ if $q \in]1, 2[$, and choosing $\tilde\theta = 1 + \frac{2}{r_1} - \frac{2}{q}$, the gradient of the mild solution y satisfies the regularity estimate:

$$(3.14) \qquad\qquad \|Dy\|_{L^{r_1}(H^{\tilde\theta,p}(\Omega))} \leq \tilde C\|f\|_{L^q(L^p(\Omega))}.$$

with a constant $\tilde C = \tilde C(p,q,r_1)$.

Remark 3.9. If we take into account Proposition 3.4, we find for $p \geq 2$, with $r \in]q, \infty]$ and $\theta = 2 + \frac{2}{r} - \frac{2}{q}$, the estimate:

$$\|y\|_{L^r(W^{\theta,p}(\Omega))} \leq C(p,q,r)\|f\|_{L^q(L^p(\Omega))},$$

while for $1 < p < 2$ and $\theta' < 2 + \frac{2}{r} - \frac{2}{q}$ one has

$$\|y\|_{L^r(W^{\theta',p}(\Omega))} \leq C(p,q,r,\theta')\|f\|_{L^q(L^p(\Omega))}.$$

Moreover, for $r_1 \in]q, \infty[$ if $q \geq 2$ and $r_1 \in]q, \frac{2q}{2-q}]$ if $q \in]1, 2[$, with $\tilde\theta = 1 + \frac{2}{r_1} - \frac{2}{q}$, when $p \geq 2$, one has the estimate

$$\|Dy\|_{L^{r_1}(W^{\tilde\theta,p}(\Omega))} \leq \tilde C(p,q,\tilde r)\|f\|_{L^q(L^p(\Omega))}$$

while when $p \in]1, 2[$, with $\theta' < 1 + \frac{2}{r_1} - \frac{2}{q}$ we have

$$\|Dy\|_{L^{r_1}(W^{\theta',p}(\Omega))} \leq \tilde C(p,q,\tilde r)\|f\|_{L^q(L^p(\Omega))}.$$

Using the Sobolev embeddings for $H^{s,p}(\Omega), W^{s,p}(\Omega)$ with $s = \theta$ or $s = \tilde{\theta}$, in the framework of Proposition 3.8 and Remark 3.11, we find the following estimates for the mild solution to (3.6):

Corollary 3.10. *With $r \in]q, \infty[$ and $\theta = 2 + \frac{2}{r} - \frac{2}{q}$ we have the estimates:*

- *For $\theta p \leq n$, choosing $\tilde{p} \leq \frac{np}{n-\theta p}$ if $\theta p < n$ and choosing arbitrarily $\tilde{p} \in [p, \infty[$ if $\theta p = n$, one has*

$$\|y\|_{L^r(L^{\tilde{p}}(\Omega)))} \leq C(p, q, r, \tilde{p})\|f\|_{L^q(L^p(\Omega))};$$

- *If $\theta p > n$, then $y \in L^r(C^{r,\alpha}(\overline{\Omega}))$ with $\alpha \in]0, 1], k \in \{0, 1\}, k + \alpha = \theta - \frac{n}{p}$ and*

$$\|y\|_{L^r(C^{k,\alpha}(\overline{\Omega})))} \leq C(p, q, r, \tilde{p})\|f\|_{L^q(L^p(\Omega))}.$$

Moreover, for $r_1 \in]q, \infty[$ if $q \geq 2$ and $r_1 \in]q, \frac{2q}{2-q}]$ if $q \in]1, 2[$, denoting by $\tilde{\theta} = 1 + \frac{2}{r_1} - \frac{2}{q}$ we have the following estimates for the gradient of the solution:

- *For $\tilde{\theta}p \leq n$, choosing $\tilde{p}_1 \leq \frac{np}{n-\tilde{\theta}p}$ if $\tilde{\theta}p < n$ and choosing arbitrarily $\tilde{p}_1 \in [p, \infty[$ if $\tilde{\theta}p = n$, one has*

$$\|Dy\|_{L^{r_1}(L^{\tilde{p}_1}(\Omega)))} \leq C(p, q, r_1, \tilde{p}_1)\|f\|_{L^q(L^p(\Omega))};$$

- *If $\tilde{\theta}p > n$, then $y \in L^{r_1}(C^{0,\alpha_1}(\overline{\Omega}))$ with $\alpha_1 \in]0, 1[, \alpha_1 = \tilde{\theta} - \frac{n}{p}$ and*

$$\|Dy\|_{L^{r_1}(C^{0,\alpha_1}(\Omega)))} \leq C(p, q, r_1, \tilde{p})\|f\|_{L^q(L^p(\Omega))}.$$

Remark 3.11. In the above considerations we studied regularity for solutions to parabolic problems with null boundary conditions and zero initial data. If we want to recover Sobolev embedding results without imposing boundary conditions, we proceed as follows. Take $\tilde{\Omega}$, some bounded domain with smooth boundary $\Omega \subset\subset \tilde{\Omega}$ and a continuous extension operator $E : W^{2,p}(\Omega) + L^p(\Omega) \to W^{2,p} \cap W_0^{1,p}(\tilde{\Omega}) + L^p(\tilde{\Omega})$. For some $u \in W_{p,q}^{2,1}(\Omega \times]0, T[)$ we have $Eu \in W_{p,q}^{2,1}(\tilde{\Omega} \times]0, T[)$. Extend now Eu by reflection to $\tilde{\Omega} \times]-T, T[$. Denote it by $\tilde{u}(x, t) = Eu(x, -t), t \in] - T, T[$ and this function belongs to $W_{p,q}^{2,1}(\tilde{Q}), \tilde{Q} = \tilde{\Omega} \times] - T, T[$. Denote by $P(D)$ the parabolic operator $P(D) = D_t - \Delta$ and take a function $\eta \in C^\infty(] - T, T[), \eta(t) = 0$ for $t < -\frac{T}{2}$ and $\eta(t) = 1$ for $t > -\frac{T}{4}$. We have

$$P(D)(\eta\tilde{u}) = \eta P(D)\tilde{u} + \eta'(t)\tilde{u}.$$

If we apply regularity estimates in $L^r(L^{\tilde{p}})$ in terms of right hand side in $L^q(L^p)$ (see Proposition 3.8), we find that

$$\|u\|_{L^r(L^{\tilde{p}}(\Omega))} \leq C\|\eta\tilde{u}\|_{L^r(-T,T;L^{\tilde{p}}(\tilde{\Omega}))} \leq C\|P(D)(\eta\tilde{u})\|_{L^q(-T,T;L^p(\tilde{\Omega}))} \leq$$

$$\leq C(\|\tilde{u}\|_{W_{p,q}^{2,1}(\tilde{Q})} + \|\tilde{u}\|_{L^q(L^p(\tilde{\Omega}))}) \leq C_1\|u\|_{W_{p,q}^{2,1}(Q)}.$$

Remark 3.12. Observe that if $u \in W_p^{2,1}(Q)$ with $p < \frac{(n+2)}{2}$ and considering the remark above with $q = p$, we may take $r = \frac{(n+2)p}{n+2-2p}$. Correspondingly, $\theta = \frac{2n}{n+2}$ and $\theta p < n$. Observing that $\tilde{p} := \frac{np}{n-\theta p} = r$, we find from Corollary 3.10 that

$$\|u\|_{L^{\frac{(n+2)p}{n+2-2p}}(Q)} \leq C(p)\|u\|_{W_p^{2,1}(Q)}.$$

Moreover, if $p < n+2$ and considering the remark above with $q = p$, we may take $r_1 = \frac{(n+2)p}{n+2-p}$. Correspondingly, $\tilde{\theta} = \frac{n}{n+2}$ and $\tilde{\theta} p < n$. Observing that $\tilde{p} := \frac{np}{n-\tilde{\theta} p} = r_1$, we find from Corollary 3.10 that

$$\|Du\|_{L^{\frac{(n+2)p}{n+2-p}}(Q)} \leq C(p)\|u\|_{W_p^{2,1}(Q)}.$$

One may see that we recovered the Sobolev type embedding in Theorem 2.2.

Concerning Sobolev embeddings for $W_{p,q}^{2,1}(Q)$ spaces we obtain from Proposition 3.8 and the Remark 3.11 the following result:

Theorem 3.13. *Consider* $u \in W_{p,q}^{2,1}(Q)$.
Then $u \in Z_1$ *where*

$$Z_1 = \begin{cases} L^r(L^{\tilde{p}}(\Omega)), \ r \in [q, \infty], \tilde{p} \leq \frac{np}{n-(2+\frac{2}{r}-\frac{2}{q})p}, & \text{if } (2+\frac{2}{r}-\frac{2}{q})p < n, \\ L^r(L^{\tilde{p}}(\Omega)), \ r \in [q, \infty], \tilde{p} \in [p, \infty[, & \text{if } (2+\frac{2}{r}-\frac{2}{q})p = n, \\ L^r(C^{k,\alpha}(\overline{\Omega})), \ \alpha \in]0,1], k \in \{0,1\}, & k+\alpha = 2+\frac{2}{r}-\frac{2}{q}-\frac{n}{p}, \\ & \text{if } (2+\frac{2}{r}-\frac{2}{q})p > n \end{cases}$$

and there exists $C = C(p,q,r,\tilde{p})$, *respectively* $C = C(p,q,r)$ *in the third case, such that*

$$\|u\|_{Z_1} \leq C\|u\|_{W_{p,q}^{2,1}(Q)}.$$

Moreover, $Du \in Z_2$ *where*

$$Z_2 = \begin{cases} L^{r_1}(L^{\tilde{p}_1}(\Omega)), \ r_1 \in [q, \infty], \tilde{p}_1 \leq \frac{np}{n-(1+\frac{2}{r_1}-\frac{2}{q})p}, & \text{if } (1+\frac{2}{r_1}-\frac{2}{q})p < n, \\ L^{r_1}(L^{\tilde{p}_1}(\Omega)), \ r_1 \in [q, \infty], \tilde{p}_1 \in [p, \infty[, & \text{if } (1+\frac{2}{r_1}-\frac{2}{q})p = n, \\ L^{r_1}(C^{0,\alpha}(\overline{\Omega})), \ \alpha \in]0,1], \alpha = 1+\frac{2}{r_1}-\frac{2}{q}-\frac{n}{p} & \text{if } (1+\frac{2}{r_1}-\frac{2}{q})p > n \end{cases}$$

and there exists $C = C(p,q,r_1,\tilde{p}_1)$, *respectively* $C = C(p,q,r_1)$ *in the third case, such that*

$$\|Du\|_{Z_2} \leq C\|u\|_{W_{p,q}^{2,1}(Q)}.$$

Consequences of temporal regularity. An immediate consequence of Theorem 3.7, considering that $D(A^\gamma) \subset H^{2\gamma,p}(\Omega)$, is the following:

Proposition 3.14. *Let* $f \in L^q(L^p(\Omega))$, $\gamma \in]0, \frac{q-1}{q}[$, *then the solution* y *to* (3.6) *given by* (3.7) *belongs to* $C^{0,\frac{q-1}{q}-\gamma}(H^{2\gamma,p}(\Omega))$ *and there exists* $C = C(p,q,\gamma)$ *such that*

$$\|y\|_{C^{0,\frac{q-1}{q}-\gamma}(H^{2\gamma,p}(\Omega))} \leq C\|f\|_{L^q(L^p(\Omega))}.$$

Now, by the previous results, taking into account Remark 3.11 and isotropic Sobolev embedding from Theorem 2.1, we obtain the following Morrey type result for $W_{p,q}^{2,1}$ spaces:

Theorem 3.15. *For $p, q \in]1, \infty[$, suppose there exists $\gamma \in]0, \frac{q-1}{q}[$ with $2\gamma - \frac{n}{p} > 0$ not an integer. Then the space $W_{p,q}^{2,1}(Q)$ is continuously embedded in $C^{0, \frac{q-1}{q} - \gamma}(C^{k,\alpha}(\overline{\Omega}))$ where $k \in \{0, 1\}$, $\alpha \in]0, 1[$, $k + \alpha = 2\gamma - \frac{n}{p}$.*

Remark 3.16. Suppose $p > nq' = \frac{nq}{q-1}$ and denote by $\alpha_0 = \frac{1}{q'} - \frac{n}{p} \in (0, 1)$. Taking first $\gamma = \frac{q-1}{2q}$ we obtain that $W_{p,q}^{2,1}(Q)$ is continuously embedded in $C^{0,\alpha_1}(C^{0,\alpha_0}(\overline{\Omega}))$ with $\alpha_1 = \frac{1}{2q'}$. Taking $\gamma = \frac{n}{p}$ and $\alpha_2 = \gamma$ we find that $W_{p,q}^{2,1}(Q)$ is continuously embedded in $C^{0,\alpha_0}(C^{0,\alpha_2}(\overline{\Omega}))$. One may see that $C^{0,\alpha_1}(C^{0,\alpha_0}(\overline{\Omega})) \cap C^{0,\alpha_0}(C^{0,\alpha_2}(\overline{\Omega}))$ is continuously embedded in the space of Hölder continuous functions $C^{0,\alpha_0}(\overline{Q})$. This conclusion is in the spirit of the result established in [27].

Remark 3.17. Theorem 3.15 has as consequence the Morrey type embedding in Theorem 2.2. Indeed, if we take $p = q > \frac{n+2}{2}$ and $\alpha \in]0, 2 - \frac{n+2}{p}[$ not an integer, we find, by choosing $\gamma = \frac{\alpha}{2} + \frac{n}{2p}$ and observing that $\alpha_1 := \frac{1}{p'} - \gamma > 0$, that $W_p^{2,1}(Q)$ is continuously embedded in $C^{0,\alpha_1}(C^{0,\alpha}(\overline{\Omega}))$.

With the same α, choose $\gamma = \frac{1}{p'} - \frac{\alpha}{2}$. Observe that $\gamma \in]\frac{n}{2p}, p'[$ and with $\alpha_2 = 2\gamma - \frac{n}{p}$ we find that $W_p^{2,1}(Q)$ is continuously embedded in $C^{0,\frac{\alpha}{2}}(C^{0,\alpha_2}(\overline{\Omega}))$. The intersection of $C^{0,\alpha_1}(C^{0,\alpha}(\overline{\Omega}))$ and $C^{0,\frac{\alpha}{2}}(C^{0,\alpha_2}(\overline{\Omega}))$ is continuously embedded in $C^{\alpha,\frac{\alpha}{2}}(\overline{Q})$ and we recover the corresponding conclusion in Theorem 2.2.

Gagliardo-Nirenberg type inequalities involving anisotropic Sobolev spaces. One may easily use Theorem 3.13 to obtain interpolation inequalities of Gagliardo type between spaces $W_{p,q}^{2,1}(Q)$ and $L^\sigma(L^\tau(\Omega))$, with $p, q, \sigma, \tau \in]1, \infty[$. If $W_{p,q}^{2,1}(Q) \subset L^r(L^{\tilde{p}})(\Omega)$ with continuous injections and $u \in W_{p,q}^{2,1}(Q) \cap L^\sigma(L^\tau(\Omega))$, then $u \in [L^r(L^{\tilde{p}}(\Omega)), L^\sigma(L^\tau(\Omega))]_\theta, \theta \in]0, 1[$ and satisfies the inequality

$$\|u\|_{L^{\sigma_\theta}(L^{\tau_\theta}(\Omega))} \leq C(\theta, p, q, \sigma, \tau)\|u\|_{W_{p,q}^{2,1}(Q)}^{1-\theta}\|u\|_{L^\sigma(L^\tau(\Omega))}^\theta,$$

where $\frac{1}{\sigma_\theta} = \frac{\theta}{\sigma} + \frac{1-\theta}{r}$ and $\frac{1}{\tau_\theta} = \frac{\theta}{\tau} + \frac{1-\theta}{\tilde{p}}$.

4. CARLEMAN INEQUALITIES IN $L^q(L^p(\Omega))$ FOR $q, p \geq 2$

In what follows, we are interested to obtain Carleman estimates in $L^q(L^p)$ spaces, when $q, p \geq 2$.

We recall first the statement of L^2 global Carleman estimates:

Let $\omega \subset\subset \Omega$. One needs (and existence is guaranteed, see [17]) an auxiliary function ψ with the following properties:

$$\psi_0 \in C^2(\overline{\Omega}),\, 0 < \psi_0 \text{ in } \Omega, \quad \psi_0|_{\partial\Omega} = 0, \quad \{x \in \overline{\Omega} : |\nabla\psi_0(x)| = 0\} \subset\subset \omega.$$

Denote by

$$(4.1) \qquad\qquad\qquad \psi := \psi_0 + K,$$

for a positive constant $K > 0$ which is fixed such that $\frac{\sup\psi}{\inf\psi} < \delta$ small enough (see [16]). Introduce also, for parameters $s, \lambda > 0$ the auxiliary functions:

$$(4.2) \qquad \varphi(t,x) := \frac{e^{\lambda\psi(x)}}{t(T-t)}, \quad \alpha(t,x) := \frac{e^{\lambda\psi(x)} - e^{1.5\lambda\|\psi\|_{C(\overline{\Omega})}}}{t(T-t)}.$$

The choice of K above is needed in order to have uniform estimates with respect to λ, with $C = C(T)$:

$$\varphi_t \le C\varphi^2,\, |\alpha_t| \le C\varphi^2,\, |\alpha_{tt}| \le C\varphi^3.$$

Following the strategy in [17], [16] one obtains in fact a family of Carleman estimates with general powers for the weight functions φ (see [37] and also [24]):

Lemma 4.1. *(Carleman estimates with general weights) Let $m \in \mathbb{R}$, then there exist $\lambda_0 = \lambda_0(\Omega, \omega, m), s_0 = s_0(\Omega, \omega, m), C = C(\Omega, \omega, m) > 0$ such that, for any $\lambda \ge \lambda_0, s \ge s_0$, the following inequality holds:*

$$\int_Q [(s\varphi)^{m-1}\lambda^m(|D_t y|^2 + |D^2 y|^2) + s^{m+1}\lambda^{m+2}\varphi^{m+1}|Dy|^2]e^{2s\alpha}dxdt$$

$$(4.3) \quad + \int_Q s^{m+3}\lambda^{m+4}\varphi^{m+3}|y|^2 e^{2s\alpha}dxdt$$

$$\le C\int_{[0,T]\times\omega} s^{m+3}\lambda^{m+4}\varphi^{m+3}|y|^2 e^{2s\alpha}dxdt + \int_Q s^m\lambda^m\varphi^m f^2 e^{2s\alpha}dxdt$$

for all $y \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega))$ solution of (1.1) with $f \in L^2(Q)$.

Using a bootstrap argument based on the above result and the regularizing effect of the parabolic flow, we obtain a family of Carleman estimates in spaces $L^q(L^p)$.

Let us now construct the following sequences: $\{q_j\}_{j\in\mathbb{N}}, q_0 = 2, q^j = q, \forall j \in \mathbb{N}^*$ and $\{p_j\}_{j\in\mathbb{N}}$ with $p_0 = 2, p_1 = \frac{2nq}{nq-4}$ and for $j \ge 2$ we define inductively

$$(4.4) \qquad p_j := \begin{cases} \dfrac{np_{j-1}}{n - \left(1 + \frac{2}{q_j} - \frac{2}{q_{j-1}}\right)p_{j-1}} = \dfrac{np_{j-1}}{n - p_{j-1}}, & \text{if } p_{j-1} < N, \\ 2p_{j-1}, & \text{if } p_{j-1} \ge N. \end{cases}$$

The sequence $\{p_j\}_{j\in\mathbb{N}}$ is increasing and $p_j \to +\infty$.

Define $\tilde{p}_j = \min\{p_j, p\}$ and take m the first index such that $\tilde{p}_m = p$. Observe that by Corollary 3.10 (see also the Theorem 3.13 for explicit orders

of integration), the solutions to parabolic problem (1.1) with null initial condition ($y_0 = 0$) satisfy for $j = 2, \ldots, m$ the estimate

$$(4.5) \quad \|y\|_{L^{q_j}(L^{\tilde{p}_j}(\Omega))} + \|Dy\|_{L^{q_j}(L^{\tilde{p}_j}(\Omega))} \leq C(\tilde{p}_{j-1}, q_{j-1}))\|f\|_{L^{q_{j-1}}(L^{\tilde{p}_{j-1}}(\Omega))}.$$

We define the sequence $\{k_j\}_{j\in\mathbb{N}}$, $k_j := k_{j-1} - 2$ with $k_0 \in \mathbb{R}$ and we denote by $w_j := \varphi^{k_j} y e^{s\alpha}$.

We write now the problem satisfied by the new variables w_j, with $j = \overline{1, m}$ using (1.1). By standard computations we find:

- $D_t(w_j) = O(s)w_{j-1} + \varphi^{k_j} D_t y e^{s\alpha}$
- $D_k(w_j) = O(s\lambda\varphi^{-1})w_{j-1} + \varphi^{k_j} D_k y e^{s\alpha}$
- $D_l(a_{kl} D_k w_j) = O(s^2\lambda^2)w_{j-1} + O(s\lambda\varphi^{-1})D_k w_{j-1} + O(s\lambda\varphi^{-1})D_l w_{j-1} + \varphi^{k_j} D_j(a_{kl} D_k y)e^{s\alpha}$.

Then, the problem takes form
(4.6)
$$\begin{cases} D_t w_j + L w_j = \varphi^{k_j} f e^{s\alpha} + O(s^2\lambda^2 w_{j-1}) + O(s\lambda D w_{j-1}), & \text{in } (0, T) \times \Omega \\ w_j = 0, & \text{on } (0, T) \times \partial\Omega, \\ w_j(0, \cdot) = 0 & \text{in } \Omega. \end{cases}$$

Since w_j satisfies an equation of type (1.1) with $w_j(0, \cdot) = 0$ in Ω, we have an inequality of type (4.5)

$$\|w_j\|_{L^{q_j}(L^{\tilde{p}_j})} + \frac{1}{s\lambda}\|Dw_j\|_{L^{q_j}(L^{\tilde{p}_j})} \leq \|w_j\|_{L^{q_j}(L^{\tilde{p}_j})} + \|Dw_j\|_{L^{q_j}(L^{\tilde{p}_j})} \leq$$

$$(4.7) \quad \leq C\left[s^2\lambda^2\|w_{j-1}\|_{L^{q_{j-1}}(L^{\tilde{p}_{j-1}})} + s\lambda\|Dw_{j-1}\|_{L^{q_{j-1}}(L^{\tilde{p}_{j-1}})}\right.$$

$$\left. + \|\varphi^{k_{j-1}} f e^{s\alpha}\|_{L^{q_{j-1}}(L^{\tilde{p}_{j-1}})}\right].$$

where, for the first inequality we choose $s\lambda > 1$.

By a standard telescopic summation procedure, after multiplying each equation in (4.7) with $(Cs^2\lambda^2)^{m-j}$ respectively and recording that $\tilde{p}_0 = q_0 = 2$, $\tilde{p}_m = p$, $q_m = q$, we obtain

$$(4.8) \quad \begin{aligned} &\|w_m\|_{L^q(L^p)} + \|Dw_m\|_{L^q(L^p)} \leq \\ &\leq C\left[s^{2m}\lambda^{2m}\|w_0\|_{L^2(Q)} + s^{2m-1}\lambda^{2m-1}\|Dw_0\|_{L^2(Q)} + E_{k_0}(f)\right], \end{aligned}$$

where we denoted by

$$E_{k_0}(f) = \sum_{j=1}^m s^{2(m-j)}\lambda^{2(m-j)}\|\varphi^{k_{j-1}} f e^{s\alpha}\|_{L^{q_{j-1}}(L^{\tilde{p}_{j-1}})}.$$

Now we write w_0, Dw_0 in terms of y, Dy and we find in the right side of (4.8) terms involving $\varphi^{k_0+1}ye^{s\alpha}$, $\varphi^{k_0}Dye^{s\alpha}$. Using the L^2 Carleman inequality (4.3), we obtain

(4.9)
$$\|w_m\|_{L^q(L^p)} + \|Dw_m\|_{L^q(L^p)}$$
$$\leq C[s^{2m}\lambda^{2m}\|\varphi^{k_0+1}ye^{s\alpha}\|_{L^2(Q_\omega)} + s^{2m-\frac{3}{2}}\lambda^{2m-2}\|\varphi^{k_0-\frac{1}{2}}fe^{s\alpha}\|_{L^2(Q)} + E_{k_0}(f)]$$
$$\leq C[s^{2m}\lambda^{2m}\|\varphi^{k_0+1}ye^{s\alpha}\|_{L^2(Q_\omega)} + \tilde{E}_{k_0}(f)],$$

where

$$\tilde{E}_{k_0}(f) = E_{k_0}(f) + s^{2m-\frac{3}{2}}\lambda^{2m-2}\|\varphi^{k_0-\frac{1}{2}}fe^{s\alpha}\|_{L^2(Q)}$$

$$\leq Cs^{2m-\frac{3}{2}}\lambda^{2m-2}\|\varphi^{k_0-\frac{1}{2}}fe^{s\alpha}\|_{L^q(L^p)}.$$

Because $Dw_m = O(s\lambda\varphi^{k_m+1}y^{s\alpha}) + \varphi^{k_m}Dye^{s\alpha}$, we obtain

(4.10)
$$\|\varphi^{k_m}ye^{s\alpha}\|_{L^q(L^p)} + s^{-1}\lambda^{-1}\|\varphi^{k_m-1}Dye^{s\alpha}\|_{L^q(L^p)}$$
$$\leq C[s^{2m}\lambda^{2m}\|\varphi^{k_0+1}ye^{s\alpha}\|_{L^2(Q_\omega)} + s^{2m-\frac{3}{2}}\lambda^{2m-2}\|\varphi^{k_0-\frac{1}{2}}fe^{s\alpha}\|_{L^q(L^p)}],$$

which gives the following theorem

Theorem 4.2. *Let $f \in L^q(L^p(\Omega)), p, q \in [2, \infty[$ and $k_0 \in \mathbb{R}$. Then there exist $m = m(q, p) \in \mathbb{N}, \lambda_0 = \lambda_0(p, q, k_0), s_0 = s_0(p, q, k_0)$ and $C = C(p, q, k_0) > 0$ such that, for any $\lambda \geq \lambda_0, s \geq s_0$, the following inequality holds:*

(4.11)
$$\|\varphi^{k_0-2m}ye^{s\alpha}\|_{L^q(L^p(\Omega))} + s^{-1}\lambda^{-1}\|\varphi^{k_0-2m-1}Dye^{s\alpha}\|_{L^q(L^p(\Omega))}$$
$$\leq C[s^{2m}\lambda^{2m}\|\varphi^{k_0+1}ye^{s\alpha}\|_{L^2(Q_\omega)} + s^{2m-\frac{3}{2}}\lambda^{2m-2}\|\varphi^{k_0-\frac{1}{2}}fe^{s\alpha}\|_{L^q(L^p(\Omega))}]$$
$$\leq C[s^{2m}\lambda^{2m}\|\varphi^{k_0+1}ye^{s\alpha}\|_{L^q(L^p(\omega))} + s^{2m-\frac{3}{2}}\lambda^{2m-2}\|\varphi^{k_0-\frac{1}{2}}fe^{s\alpha}\|_{L^q(L^p(\Omega))}].$$

Remark 4.3. The above regularity argument may be further used in order to have estimates also for D^2y and D_ty in $L^q(L^p)$ spaces or, if q, p are big enough, estimates for y in anisotropic Hölder spaces.

Explicit dependence on T of constant C appearing in Carleman inequality (4.11) may be obtained by using the results in [16]. Indeed, we perform here only a finite number of boot-strap arguments involving constants from Sobolev embeddings so, for fixed principal part of operator L, one may write for the constant in (4.11)

$$C = \exp\left[K(\Omega, \omega, \|b_k, c\|_{L^\infty})\left(\frac{1}{T} + T\right)\right].$$

This kind of estimate is useful in nonlinear controllability problems (see [15],[14]), as well as in unique continuation results at initial time (see [21], [2]).

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